

C^* -algebras associated to topological Ore semigroups

S. Sundar

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Abstract

Let G be a locally compact group and $P \subset G$ be a closed Ore semigroup containing the identity element. Let $V : P \rightarrow B(\mathcal{H})$ be an anti-homomorphism such that for every $a \in P$, V_a is an isometry and the final projections of $\{V_a : a \in P\}$ commute. We study the C^* -algebra generated by $\{\int f(a)V_a da : f \in L^1(P)\}$. We show that there exists a groupoid C^* -algebra which is universal for isometric representations with commuting range projections.

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1 Introduction

It is fair to say that C^* -algebras of groups and their crossed products are the most studied C^* -algebras in the theory of operator algebras. Several authors have tried to study C^* -algebras associated to semigroups. For example, the Toeplitz algebra is the C^* -algebra associated to the additive semigroup \mathbb{N} . Recently, the theory of semigroup C^* -algebras have received renewed attention. See for example [Cun08], [Li12], [Li13] and the references therein. The notion of crossed product by semigroups has also been studied by several authors most notably by Murphy in [Mur91], [Mur94], [Mur96b] and by Exel in [Exe03]. However much of the literature focusses on discrete semigroups. In the topological direction, upto the author's knowledge, the only example studied is the Wiener-Hopf C^* -algebra. This was studied from the groupoid point of view first in [MR82] and then successively by Nica in [Nic87], [Nic90] and Hilgert and Neeb in [HN95].

Let G be a second countable locally compact group and $P \subset G$ be a closed semigroup containing the identity element. We assume that $Int(P)$ is dense in P and $PP^{-1} = G$.

Let $V : P \rightarrow B(\mathcal{H})$ be an isometric representation on a Hilbert space \mathcal{H} i.e. for $a \in P$, V_a is an isometry and $V_a V_b = V_{ba}$. For $f \in L^1(P)$, let

$$W_f := \int_{a \in P} f(a) V_a da.$$

The semigroup C^* -algebra or the Wiener-Hopf algebra, denoted $\mathcal{W}_V(P, G)$, associated to the representation V is the C^* -algebra generated by $\{W_f : f \in L^1(P)\}$. If we consider the compression of the right regular representation of G on $L^2(G)$ onto $L^2(P)$, then one obtains the usual Wiener-Hopf algebra studied in [MR82]. In general, it is much difficult to understand the structure of $\mathcal{W}_V(P, G)$. However if we assume that the final projections $\{E_a := V_a V_a^* : a \in P\}$ form a commuting family of projections then one can do better. Without this commutative assumption, the situation becomes much complicated even for the simplest case of $P := \mathbb{N} \times \mathbb{N}$ as is illustrated by Murphy in [Mur96a]. The results obtained and the organisation of the paper are described below.

From now on, we assume that the range projections commute. For $g = ab^{-1} \in G$, let $W_g := V_b^* V_a$ and E_g be the final space of W_g . It is shown in Section 3, that W_g is well-defined and $\{E_g : g \in G\}$ forms a commuting family of projections. For $f \in L^1(G)$, let $W_f := \int f(g) W_g dg$. It is not difficult to show that $\mathcal{W}_V(P, G)$ is generated by $\{W_f : f \in L^1(G)\}$. Let Ω be the spectrum of the commutative C^* -algebra generated by $\{\int f(g) E_g dg : f \in L^1(G)\}$. The map $C(\Omega) \rtimes P \ni (T, a) \rightarrow V_a^* T V_a \in C(\Omega)$ provides an action of P on Ω . In Section 4 and 5, we show that this action is injective. Let

$$\mathcal{G} := \Omega \rtimes P := \{(x, ab^{-1}, y) \in \Omega \times G \times \Omega : xa = yb\}$$

be the Deaconu-Renault groupoid where the groupoid operations are given by

$$\begin{aligned} (x, g, y)(y, h, z) &= (x, gh, z) \\ (x, g, y)^{-1} &= (y, g^{-1}, x). \end{aligned}$$

For $f \in C_c(G)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x, g, y) = f(g)$. We apply the results of [RS15] to show that $\Omega \rtimes P$ has a Haar system. We also show that there exists a surjective representation $\lambda : C^*(\mathcal{G}) \rightarrow \mathcal{W}_V(P, G)$ such that for $f \in C_c(G)$,

$$\lambda(\tilde{f}) = \int f(g) \Delta(g)^{-\frac{1}{2}} W_{g^{-1}} dg.$$

Here Δ denotes the modular function of the group. This is achieved in Sections 4-6. For the Wiener-Hopf representation, the groupoid $\Omega \rtimes P$ is the groupoid considered in [MR82].

We show in Section 7, that there exists a universal space Ω_u on which P acts such that if $V : P \rightarrow B(\mathcal{H})$ is an isometric representation with commuting range projections then there exists a representation $\lambda : C^*(\Omega_u \rtimes P) \rightarrow B(\mathcal{H})$ such that for $f \in C_c(G)$,

$$\lambda(\tilde{f}) = \int f(g) \Delta(g)^{-\frac{1}{2}} W_{g^{-1}} dg.$$

2 Preliminaries

For the convenience of the reader, we recall the essential facts from [RS15] that we need in this paper. The proofs can be found in [RS15]. Throughout this paper, G stands for a second countable, locally compact topological group and $P \subset G$ for a closed subsemigroup containing the identity element e . We also assume the following.

(C1) The group $G = PP^{-1}$, and

(C2) the interior of P in G , denoted $Int(P)$, is dense in P .

Semigroups for which (C1) is satisfied are called Ore semigroups. In this paper, we consider only semigroups with identity for which (C1) and (C2) are satisfied.

Let X be a compact Hausdorff space. A right action of P on X is a continuous map $X \times P \ni (x, a) \rightarrow xa \in X$ such that $xe = x$ and $(xa)b = x(ab)$ for $x \in X$ and $a, b \in P$. Moreover we assume that the action is injective i.e. for every $a \in P$, the map $X \ni x \rightarrow xa \in X$ is injective. Let X be a compact Hausdorff space on which P acts on the right injectively. Then the semi-direct product groupoid $X \rtimes P$ is defined as follows:

$$X \rtimes P := \{(x, g, y) \in X \times G \times X : \exists a, b \in P, \text{ such that } g = ab^{-1}, xa = yb\}.$$

The groupoid multiplication and the inversion are given by

$$\begin{aligned} (x, g, y)(y, h, z) &= (x, gh, z), \\ (x, g, y)^{-1} &= (y, g^{-1}, x). \end{aligned}$$

The map $X \rtimes P \ni (x, g, y) \rightarrow (x, g) \in X \times G$ is injective. Thus $X \rtimes P$ can be considered a subset of $X \times G$ which we do from now. Moreover $X \rtimes P$ is a closed subset of $X \times G$ and when $X \rtimes P$ is given the subspace topology, the groupoid $X \rtimes P$ becomes a topological groupoid. We denote the range and source maps by r and s respectively.

For $x \in X$, let $Q_x := \{g \in G : (x, g) \in \mathcal{G}\}$. Then $r^{-1}(x) = \{x\} \times Q_x$. Note that for $x \in X$, $Q_x \cdot P \subset Q_x$ and Q_x is closed. By Lemma 4.1 of [RS15], for $x \in X$, $Int(Q_x)$ is dense in Q_x and the boundary of Q_x has measure zero.

For $x \in X$, let λ^x be the measure on \mathcal{G} defined as follows: For $f \in C_c(\mathcal{G})$,

$$\int f d\lambda^x = \int f(x, g) 1_{Q_x}(g) dg.$$

Here dg denotes the left Haar measure on G . In [RS15], it is shown that the groupoid $\mathcal{G} := X \rtimes P$ admits a Haar system if and only if the map $X \times \text{Int}(P) \ni (x, a) \rightarrow xa \in X$ is open. In this case, the measures $(\lambda^x)_{x \in X}$ form a Haar system. We will use only this Haar system if $X \rtimes P$ admits one.

Suppose that $\mathcal{G} := X \rtimes P$ admits a Haar system. Then the action of P on X can be dilated to an action of G . That is there exists a locally compact Hausdorff space Y on which G acts on the right and a continuous P -equivariant injection $i : X \rightarrow Y$ such that

1. the set $X_0 := i(X)\text{Int}(P)$ is open in Y , and
2. $Y = \bigcap_{a \in P} i(X)a^{-1} = \bigcap_{a \in \text{Int}(P)} X_0 a^{-1}$.

Moreover the space Y is unique up to a G -equivariant homeomorphism. We will identify X as a subspace of Y via the injection i and will suppress the notation i . Also the groupoid \mathcal{G} is isomorphic to the reduction $(Y \rtimes G)|_X$. With this notation, note that for $x \in X$, $1_{Q_x}(g) = 1_X(xg)$. Also we leave it to the reader to check that $1_{\text{Int}(Q_x)}(g) = 1_{X_0}(xg)$.

For $f \in C_c(G)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x, g) = f(g)$. We also need the following proposition. The proof is a line by line imitation of that of Proposition 3.5 of [MR82]. Hence we omit the proof. See also [RS15] for some remarks concerning the proof.

Let $\mathcal{G} := X \rtimes P$ and assume that it has a Haar system.

Proposition 2.1 *For $f \in C_c(G)$, let $\hat{f} \in C(X)$ be defined by $\hat{f}(x) = \int f(g) 1_X(xg) dg$. Suppose that family $\{\hat{f} : f \in C_c(G)\}$ separates points of X . Then the $*$ -algebra generated by $\{\tilde{f} : f \in C_c(G)\}$ is dense in $C_c(\mathcal{G})$ where $C_c(\mathcal{G})$ is given the inductive limit topology. As a consequence, $C^*(\mathcal{G})$ is generated by $\{\tilde{f} : f \in C_c(G)\}$.*

3 Isometric representations with commuting range projections

Definition 3.1 *A map $V : P \rightarrow B(\mathcal{H})$ is called an isometric representation of P on the Hilbert space \mathcal{H} if*

- (1) *the maps $P \ni a \rightarrow V_a$ and $P \ni a \rightarrow V_a^*$ are strongly continuous,*

(2) for $a \in P$, V_a is an isometry, and

(3) for $a, b \in P$, $V_a V_b = V_{ba}$.

For $a \in P$, let $E_a := V_a V_a^*$. If $\{E_a : a \in P\}$ is a commuting family of projections, we say that V has commuting range projections.

In the next example, we recall the Wiener-Hopf representation or the regular representation. The C^* -algebra associated to the Wiener-Hopf representation has been studied by several authors. See the papers [MR82], [HN95] and the references therein.

Example 3.2 Consider the Hilbert space $L^2(G)$ and consider $L^2(P)$ as a closed subspace of $L^2(G)$. For $\xi \in L^2(P)$ and $a \in P$, let $V_a(\xi)$ be defined as follows:

$$V_a(\xi)(x) := \begin{cases} \xi(xa^{-1})\Delta(a)^{-\frac{1}{2}} & \text{if } x \in Pa \\ 0 & \text{if } x \notin Pa. \end{cases}$$

Here Δ denotes the modular function of the group G . Then the map $a \in P \rightarrow V_a \in B(L^2(P))$ is an isometric representation with commuting range projections. Note that for $a \in P$, the range of V_a is $L^2(Pa)$.

Till the end of Section 7, we fix an isometric representation $V : P \rightarrow B(\mathcal{H})$ with commuting range projections. For $g = ab^{-1}$, let $W_g := V_b^* V_a$ and let $E_g := W_g W_g^*$. First we show that W_g is well defined and is a partial isometry.

Proposition 3.3 Let $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections.

(1) For $g \in G$, W_g is well defined and is a partial isometry.

(2) The family $\{E_g : g \in G\}$ forms a commuting family of projections.

(3) If $g_1 g_2^{-1} \in P$, then $E_{g_1} \leq E_{g_2}$.

(4) The map $G \ni g \rightarrow W_g \in B(\mathcal{H})$ is strongly continuous.

(5) For $g, h \in G$, $W_g W_h = E_g W_{hg}$.

Proof. Suppose $g = a_1 b_1^{-1} = a_2 b_2^{-1}$. Then $a_1^{-1} a_2 = b_1^{-1} b_2$. Since $PP^{-1} = G$, there exists $\alpha_1, \alpha_2 \in P$ such that $a_1^{-1} a_2 = \alpha_1 \alpha_2^{-1} = b_1^{-1} b_2$. Then $a_1 \alpha_1 = a_2 \alpha_2$ and $b_1 \alpha_1 = b_2 \alpha_2$.

Now observe that

$$\begin{aligned}
V_{b_1}^* V_{a_1} &= V_{b_1}^* V_{\alpha_1}^* V_{\alpha_1} V_{a_1} \\
&= V_{b_1 \alpha_1}^* V_{a_1 \alpha_1} \\
&= V_{b_2 \alpha_2}^* V_{a_2 \alpha_2} \\
&= V_{b_2}^* V_{\alpha_2}^* V_{\alpha_2} V_{a_2} \\
&= V_{b_2}^* V_{a_2}.
\end{aligned}$$

This proves that W_g is well defined. Let $E_g := W_g W_g^*$. If $g = ab^{-1}$, then $E_g = V_b^* E_a V_b$ which is self adjoint. Now note that

$$\begin{aligned}
E_g^2 &= V_b^* E_a V_b V_b^* E_a V_b \\
&= V_b^* E_a E_b E_a V_b \\
&= V_b^* E_b E_a^2 V_b \quad (\text{Since } E_a \text{ and } E_b \text{ commute}) \\
&= V_b^* E_a V_b \\
&= E_g.
\end{aligned}$$

Thus E_g is a projection. This proves (1).

Let $g_1, g_2 \in G$ be given. Write $g_1 = a_1 b_1^{-1}$ and $g_2 = a_2 b_2^{-1}$ with $a_i, b_i \in P$. Choose $\alpha_1, \alpha_2 \in P$ such that $b_1 \alpha_1 = b_2 \alpha_2$. Let $a'_i = a_i \alpha_i$ and $b'_i = b_i \alpha_i$ for $i = 1, 2$. Then $g_i = a'_i (b'_i)^{-1}$ for $i = 1, 2$. But now $b'_1 = b'_2$. Thus

$$\begin{aligned}
E_{g_1} E_{g_2} &= V_{b'_1}^* E_{a'_1} V_{b'_1} V_{b'_1}^* E_{a'_2} V_{b'_1} \\
&= V_{b'_1}^* E_{a'_2} V_{b'_1} V_{b'_1}^* E_{a'_1} V_{b'_1} \\
&= E_{g_2} E_{g_1}.
\end{aligned}$$

This proves (2).

Suppose $g_1 g_2^{-1} = a$ for some $a \in P$. Write $g_2 = bc^{-1}$. Then $g_1 = (ab)c^{-1}$. Then

$$\begin{aligned}
E_{g_1} &= V_c^* V_{ab} V_{ab}^* V_c \\
&= V_c^* V_b (V_a V_a^*) V_b^* V_c \\
&\leq V_c^* V_b V_b^* V_c \\
&\leq E_{g_2}.
\end{aligned}$$

This proves (3).

Note that the map $Int(P) \times Int(P) \ni (a, b) \rightarrow ab^{-1} \in G$ is surjective and open. Thus G is the quotient of $Int(P) \times Int(P)$. Since multiplication is strongly continuous on the

unit ball of $B(\mathcal{H})$, it follows that the map $Int(P) \times Int(P) \ni (a, b) \rightarrow V_b^* V_a \in B(\mathcal{H})$ is strongly continuous. As a consequence, it follows that the map $G \ni g \rightarrow W_g \in B(\mathcal{H})$ is strongly continuous. This proves (4).

Let $g, h \in G$ be given. Write $g = ab^{-1}$ and $h = cd^{-1}$ with $a, b \in P$. Choose $\alpha, \beta \in P$ such that $d\beta = a\alpha$. Now note that $g = (a\alpha)(b\alpha)^{-1}$ and $h = (c\beta)(d\beta)^{-1}$. Thus we can write g and h as $g = a_1 b_1^{-1}$ and $h = c_1 a_1^{-1}$. Now calculate as follows

$$\begin{aligned} W_g W_h &= V_{b_1}^* V_{a_1} V_{a_1}^* V_{c_1} \\ &= V_{b_1}^* E_{a_1} V_{b_1} V_{b_1}^* V_{c_1} \text{ (Since } E_{a_1} \text{ and } E_{b_1} \text{ commute)} \\ &= E_g W_{hg} \end{aligned}$$

This completes the proof. □

For $f \in L^1(G)$, the “Wiener-Hopf” operator with symbol f is defined as

$$W_f := \int f(g) W_g dg.$$

We want to describe the C^* -algebra, denoted $\mathcal{W}_V(P, G)$, generated by $\{W_f : f \in L^1(G)\}$. We suppress the subscript V and simply denote $\mathcal{W}_V(P, G)$ by $\mathcal{W}(P, G)$ (at least till the end of Section 7.)

Remark 3.4 *One can show that $\mathcal{W}(P, G)$ is generated by $\{\int f(a) V_a da : f \in L^1(P)\}$. The proof is similar to that of Proposition 2.2 of [RS15]. Hence we omit the proof.*

First we consider a related commutative C^* -algebra. Note that by definition, for $g \in G$, $W_{g^{-1}} = W_g^*$. Moreover the map $G \ni g \rightarrow E_g = W_g W_g^*$ is strongly continuous. For $f \in L^1(G)$, let

$$E_f := \int f(g) E_g dg.$$

Let

$$\mathcal{A} := C^*\{E_f : f \in L^1(G)\}.$$

Since $\{E_g : g \in G\}$ forms a commuting family of projections, it follows that \mathcal{A} is a commutative C^* -subalgebra of $B(\mathcal{H})$. Note that $E_g = 1$ if $g \in P^{-1}$. If $f \in L^1(P^{-1})$, then $E_f = \int f(g) dg$. Thus, it follows that \mathcal{A} is a commutative unital C^* subalgebra of $B(\mathcal{H})$. Denote the spectrum of \mathcal{A} by Ω .

Let $G_n := \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}$. For $f \in C_c(G_n)$, let

$$E_f := \int f(g_1, g_2, \dots, g_n) E_{g_1} E_{g_2} \cdots E_{g_n} dg_1 dg_2 \cdots dg_n.$$

Let $\tilde{\mathcal{A}} := \bigcup_{n=1}^{\infty} \{E_f : f \in C_c(G_n)\}$. Then $\tilde{\mathcal{A}}$ forms a dense unital $*$ -subalgebra of \mathcal{A} . Also note that for every n , the map $C_c(G_n) \ni f \rightarrow E_f \in \mathcal{A}$ is continuous when $C_c(G_n)$ is given the inductive limit topology and \mathcal{A} is given the norm topology.

For $T \in B(\mathcal{H})$ and $a \in P$, let $\alpha_a(T) = V_a^* T V_a$. Clearly $\alpha_e = id$ and $\alpha_a \alpha_b = \alpha_{ab}$.

Observe that $\alpha_a(V_b^* E_c V_b) = V_a^* V_b^* E_c V_b V_a = V_{ab}^* E_c V_{ab}$. Thus $\alpha_a(E_g) = E_{ga^{-1}}$ for $g \in G$. Since the final projection $V_a V_a^*$ commutes with E_g for every $g \in G$, it follows that $\alpha_a(E_{g_1} E_{g_2}) = \alpha_a(E_{g_1}) \alpha_a(E_{g_2})$.

Proposition 3.5 *For $a \in P$, α_a leaves \mathcal{A} invariant and the map $\alpha_a : \mathcal{A} \rightarrow \mathcal{A}$ is a unital $*$ -homomorphism. Moreover for $T \in \mathcal{A}$, the map $P \ni a \rightarrow \alpha_a(T) \in \mathcal{A}$ is norm continuous.*

Proof. For $a \in P$ and $f \in C_c(G_n)$, let $\tilde{f}_a \in C_c(G_n)$ be defined by

$$\tilde{f}_a(g_1, g_2, \dots, g_n) = \Delta(a)^n f(g_1 a, g_2 a, \dots, g_n a).$$

Then for $f \in C_c(G_n)$, the map $P \ni a \rightarrow \tilde{f}_a \in C_c(G_n)$ is continuous if $C_c(G_n)$ is given the inductive limit topology.

Let $a \in P$ and $f \in C_c(G_n)$ be given. Then

$$\begin{aligned} \alpha_a(E_f) &= \int f(g_1, g_2, \dots, g_n) \alpha_a(E_{g_1} E_{g_2} \cdots E_{g_n}) dg_1 dg_2 \cdots dg_n \\ &= \int f(g_1, g_2, \dots, g_n) E_{g_1 a^{-1}} E_{g_2 a^{-1}} \cdots E_{g_n a^{-1}} dg_1 dg_2 \cdots dg_n \\ &= \int f(g_1 a, g_2 a, \dots, g_n a) \Delta(a)^n E_{g_1} E_{g_2} \cdots E_{g_n} dg_1 dg_2 \cdots dg_n \\ &= E_{\tilde{f}_a} \end{aligned}$$

Thus α_a leaves $\tilde{\mathcal{A}}$ invariant. Since $\tilde{\mathcal{A}}$ is dense in \mathcal{A} and α_a is bounded, it follows that α_a leaves \mathcal{A} invariant.

Observe that if $E_a = V_a V_a^*$ commutes with $T, S \in B(\mathcal{H})$ then $\alpha_a(TS) = \alpha_a(T) \alpha_a(S)$. By Proposition 3.3, it follows that E_a commutes with E_f for $f \in C_c(G_n)$. Thus E_a commutes with every element of \mathcal{A} . Hence $\alpha_a : \mathcal{A} \rightarrow \mathcal{A}$ is multiplicative. Clearly α_a is unital and $*$ -preserving.

For $a \in P$, $\alpha_a : \mathcal{A} \rightarrow \mathcal{A}$ is contractive. Thus it is enough to show that for $T \in \tilde{\mathcal{A}}$, the map $P \ni a \rightarrow \alpha_a(T) \in \mathcal{A}$ is continuous. Let $T = E_f$ for some $f \in C_c(G_n)$. Then $\alpha_a(T) = E_{\tilde{f}_a}$. Hence the map $P \ni a \rightarrow \alpha_a(T) = E_{\tilde{f}_a}$ is continuous as it is the composite

of the continuous maps $P \ni a \rightarrow \tilde{f}_a \in C_c(G_n)$ and $C_c(G_n) \ni h \rightarrow E_h$ where $C_c(G_n)$ is given the inductive limit topology. This completes the proof. \square

Since $\mathcal{A} = C(\Omega)$, it follows that for every $a \in P$, there exists $\phi_a : \Omega \rightarrow \Omega$ such that $F \circ \phi_a = \alpha_a(F)$ for $F \in C(\Omega)$. The condition $\alpha_a \alpha_b = \alpha_{ab}$ translates to $\phi_a \phi_b = \phi_{ba}$ for $a, b \in P$. Also $\phi_e = id$. Thus the map $\Omega \times P \ni (x, a) \rightarrow \phi_a(x) \in \Omega$ defines a right action of P on Ω . We henceforth write $\phi_a(x)$ as xa for $x \in \Omega$ and $a \in P$.

We claim that the map $\Omega \times P \ni (x, a) \rightarrow xa \in \Omega$ is continuous. Suppose $(x_n) \rightarrow x$ and $(a_n) \rightarrow a$. Let $F \in C(\Omega)$. By Proposition 3.5, it follows that $\alpha_{a_n}(F)$ converges uniformly to $\alpha_a(F)$. Since the convergence is uniform, it follows that $\alpha_{a_n}(F)(x_n)$ converges to $\alpha_a(F)(x)$. In other words, for every $F \in C(\Omega)$, $F(x_n a_n)$ converges to $F(xa)$. Hence $x_n a_n$ converges to xa .

The goal of this paper is to prove the following statements.

- (1) The right action of P on Ω is injective.
- (2) The semidirect product groupoid $\mathcal{G} := \Omega \rtimes P$ has a Haar system.
- (3) For $f \in C_c(G)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x, g) = f(g)$ for $(x, g) \in \mathcal{G}$. There exists a surjective $*$ -homomorphism $\pi : C^*(\mathcal{G}) \rightarrow \mathcal{W}(P, G)$ such that $\pi(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g) W_{g^{-1}} dg$ for $f \in C_c(G)$.

To prove the above statements, we need a better description of Ω which forms the content of the next section. We end this section with a lemma which is useful in showing that $\Omega \rtimes P$ has a Haar system.

Lemma 3.6 *Let $f \in C_c(G)$ be such that $\text{supp}(f) \subset \text{Int}(P)$. Then for $T \in \mathcal{A}$, the integral $\int_{a \in P} f(a) V_a T V_a^* da \in \mathcal{A}$.*

Proof. It is enough to prove the statement for $T \in \tilde{\mathcal{A}}$. Let $T = E_\phi$ for some $\phi \in C_c(G_n)$. For $a \in P$ and $g \in G$, $V_a^* E_{ga} V_a = E_g$. Hence $V_a E_g V_a^* = E_{ga} E_a$. Now calculate as follows

to find that

$$\begin{aligned}
& \int_{a \in P} f(a) V_a E_\phi V_a^* da \\
&= \int_{a \in P} f(a) \phi(g_1, g_2, \dots, g_n) V_a E_{g_1} E_{g_2} \cdots E_{g_n} V_a^* da \, dg_1 \, dg_2 \cdots dg_n \\
&= \int_{a \in P} f(a) \phi(g_1, g_2, \dots, g_n) E_a E_{g_1 a} E_{g_2 a} \cdots E_{g_n a} da \, dg_1 dg_2 \cdots dg_n \\
&= \int_{a \in P} \Delta(a)^{-n} f(a) \phi(g_1 a^{-1}, g_2 a^{-1}, \dots, g_n a^{-1}) E_a E_{g_1} E_{g_2} \cdots E_{g_n} da \, dg_1 \, dg_2 \cdots dg_n \\
&= E_\psi \in \tilde{\mathcal{A}}
\end{aligned}$$

where $\psi \in C_c(G_{n+1})$ is given by

$$\psi(g, g_1, g_2, \dots, g_n) = \Delta(g)^{-n} f(g) \phi(g_1 g^{-1}, g_2 g^{-1}, \dots, g_n g^{-1}).$$

This completes the proof. \square

4 What is Ω ?

We first discuss the case when G is discrete. The discrete semigroup C^* -algebras are analysed in great detail in the papers [Li12] and [Li13]. Nevertheless we discuss this case in the form that we need. This also motivates the topological case.

Let G be a discrete group and $P \subset G$ be a semigroup such that $e \in P$ and $PP^{-1} = G$. In this case, the Wiener-Hopf C^* -algebra $\mathcal{W}_V(P, G)$ is simply the C^* -algebra generated by $\{W_g : g \in G\}$ and the commutative C^* -algebra \mathcal{A} is the C^* -algebra generated by $\{E_g : g \in G\}$.

Let χ be a character of \mathcal{A} . Let us define the support of χ , denoted A_χ , as

$$A_\chi := \{g \in G : \chi(E_g) = 1\}.$$

Condition (3) of Proposition 3.3 implies that $P^{-1}A_\chi \subset A_\chi$. Since $E_g = 1$ if $g \in P^{-1}$, it follows that $P^{-1} \subset A_\chi$.

Let $\mathcal{P}(G)$ be the power set of G . Identify $\mathcal{P}(G)$ with $\{0, 1\}^G$, via the map $\mathcal{P}(G) \ni A \rightarrow 1_A \in \{0, 1\}^G$, and endow it with the product topology. The group G acts on $\mathcal{P}(G)$. The right action is given by : For $g \in G$ and $A \in \mathcal{P}(G)$, $Ag := \{ag : a \in A\}$. Clearly the map $\Omega \ni \chi \rightarrow A_\chi \in \{0, 1\}^G$ is continuous, injective and hence an embedding. We leave it to the reader to check that the above map is P -equivariant. From now, we view Ω as a subset of $\{0, 1\}^G$.

Proposition 4.1 *We have the following.*

- (1) *For $A \in \Omega$ and $a \in P$, $Aa^{-1} \in \Omega$ if and only if $a \in A$.*
- (2) *For $A \in \Omega$ and $g \in G$, $Ag \in \Omega$ if and only if $g^{-1} \in A$.*
- (3) *The action $\Omega \times P \rightarrow \Omega$ is open.*

Proof. Let $A \in \Omega$ and $a \in P$ be given. Suppose $B := Aa^{-1} \in \Omega$. Since $e \in B$, it follows that $a \in A$. Now suppose $a \in A$. Let χ be the character corresponding to A . Since $V_a^* E_{ga} V_a = E_g$, it follows that $V_a E_g V_a^* = E_a E_{ga}$. Thus the homomorphism $B(\mathcal{H}) \ni T \rightarrow V_a T V_a^* \in B(\mathcal{H})$ leaves \mathcal{A} invariant.

Let $\tilde{\chi}$ be the character on \mathcal{A} defined by $\tilde{\chi}(T) = \chi(V_a T V_a^*)$. Since $\chi(E_a) = 1$, it follows that $\tilde{\chi}$ is non-zero. Observe that for $T \in \mathcal{A}$,

$$\begin{aligned}
(\tilde{\chi}a)(T) &= \tilde{\chi}(V_a^* T V_a) \\
&= \chi(V_a V_a^* T V_a V_a^*) \\
&= \chi(E_a) \chi(T) \chi(E_a) \\
&= 1_A(a) \chi(T) 1_A(a) \\
&= \chi(T).
\end{aligned}$$

Thus $\tilde{\chi}a = \chi$. Let B be the support of $\tilde{\chi}$. Then $A = Ba$. Thus $Aa^{-1} \in \Omega$. This proves (1).

Now let $A \in \Omega$ and $g = ab^{-1} \in G$. Suppose $g^{-1} = ba^{-1} \in A$. Then $b \in Aa \in \Omega$. By (1), it follows that $Ag = Aab^{-1} \in \Omega$. Now suppose $Ag \in \Omega$. Then $A = (Ag)g^{-1}$. Since $e \in Ag$, it follows that $g^{-1} \in A$. This proves (2).

When G is discrete, $\text{Int}(P) = P$ and $\Omega P = \Omega$. Thus, by Theorem 4.3 of [RS15], to prove that the action $\Omega \times P \rightarrow \Omega$ is open, it is enough to show that Ωa is open in Ω for every $a \in P$. But note that by (1), for $a \in P$, $\Omega a = \{A \in \Omega : 1_A(a) = 1\}$ which is clearly open in Ω , as Ω has the subspace topology of $\{0, 1\}^G$. This completes the proof. \square

A consequence of Proposition 4.1 is that the semi-direct product groupoid $\Omega \rtimes P$ has a Haar system. For $g \in G$, let $\delta_g \in C_c(\Omega \times P)$ be defined by $\delta_g(x, h) = 1$ if $h = g$ and $\delta_g(x, h) = 0$ if $h \neq g$. Then it is not difficult to show that there exists a representation $\pi : C_c(\Omega \times P) \rightarrow B(\mathcal{H})$ such that $\pi(\delta_g) = W_{g^{-1}}$ for every $g \in G$. We will prove this in the topological case.

Now let us turn our attention to the topological case. Let χ be a character of the commutative C^* -algebra \mathcal{A} . The **support** of χ , denoted A_χ , is defined as follows: For

$g \in G$, $g \notin A_\chi$ if and only if there exists an open set U of G containing g such that $\chi(\int f(g)E_g dg) = 0$ for every $f \in C_c(U)$. Here $C_c(U) := \{f \in C_c(G) : \text{supp}(f) \subset U\}$. Note that A_χ is closed.

Remark 4.2 *Let χ be a character of \mathcal{A} and A be its support. Then for $g \in G$, $g \in A$ if and only if for every open set U containing g , there exists $f \in C_c(U)$ such that $f \geq 0$ and $\chi(\int f(g)E_g dg) > 0$.*

Proposition 4.3 *Let χ be a character of \mathcal{A} and let A be its support. Then*

- (1) $P^{-1} \subset A$ and $P^{-1}A \subset A$,
- (2) the interior $\text{Int}(A)$ is dense in A , and
- (3) the boundary $\partial(A)$ has measure zero.

Proof. Let $a \in \text{Int}(P)$ and U be an open set containing a^{-1} . Then $U \cap \text{Int}(P)^{-1}$ is a non-empty open set containing a^{-1} . Choose $f \in C_c(G)$ such that $\text{supp}(f) \subset U \cap \text{Int}(P)^{-1}$, $f \geq 0$ and $\int f(g)dg = 1$. Since $E_g = 1$ for $g \in P^{-1}$, it follows that $\int f(g)E_g dg = \int f(g)dg = 1$. Thus $\chi(\int f(g)E_g dg) = 1$. This proves that $a^{-1} \in A$. As a consequence, $\text{Int}(P)^{-1} \subset A$. But $\text{Int}(P)^{-1}$ is dense in P^{-1} and A is closed. Hence $P^{-1} \subset A$.

For $f \in C_c(G)$ and $g \in G$, let $L_g(f) \in C_c(G)$ be defined by $L_g(f)(x) = f(g^{-1}x)$.

Let $g \in A$ be given and $a \in P$. Let U be an open set containing $a^{-1}g$. Then aU is open and contains g . Thus there exists $f \in C_c(aU)$ such that $f \geq 0$ and $\chi(\int f(g)E_g dg) > 0$. Let $\tilde{f} = L_{a^{-1}}f$. Then $\tilde{f} \geq 0$ and $\text{supp}(\tilde{f}) \subset U$. Now

$$\begin{aligned} \int \tilde{f}(g)E_g dg &= \int f(ag)E_g dg \\ &= \int f(g)E_{a^{-1}g} dg \\ &\geq \int f(g)E_g dg \quad (\text{By Proposition 3.3}). \end{aligned}$$

Hence $\chi(\int \tilde{f}(g)E_g dg) > 0$. This implies that $a^{-1}g \in A$. Thus $P^{-1}A \subset A$. This proves (1). Statements (2) and (3) follow immediately from Lemma 4.1 of [RS15]. This completes the proof. \square

Before proceeding further, let us review the Vietoris topology. Let X be a locally compact second countable Hausdorff space and let d be a metric on X inducing the topology. Let $\mathcal{C}(X)$ be the collection of closed subsets of X . Then $\mathcal{C}(X)$, endowed with

the Vietoris topology, is compact and metrisable. We recall here the convergence of sequences of elements in $\mathcal{C}(X)$.

Let (A_n) be a sequence of closed subsets of X . Define

$$\begin{aligned}\liminf A_n &= \{x \in X : \limsup d(x, A_n) = 0\}, \text{ and} \\ \limsup A_n &= \{x \in X : \liminf d(x, A_n) = 0\}.\end{aligned}$$

Then (A_n) converges in $\mathcal{C}(X)$ if and only if $\liminf A_n = \limsup A_n$. If $\liminf A_n = \limsup A_n = A$, then A_n converges to A . Observe that if $U \subset X$ is closed then the subset $\{A \in \mathcal{C}(X) : A \cap U \neq \emptyset\}$ is open in $\mathcal{C}(X)$.

Consider $\mathcal{C}(G)$, the space of closed subsets of G , with the Vietoris topology. The group G acts on $\mathcal{C}(G)$ on the right. For $A \in \mathcal{C}(G)$ and $g \in G$, define $Ag = \{ag : a \in A\}$. Let

$$\Omega_u := \{A \in \mathcal{C}(G) : P^{-1} \subset A \text{ and } P^{-1}A \subset A\}.$$

We leave it to the reader to verify that Ω_u is a closed, and hence a compact, subset of $\mathcal{C}(G)$. Clearly Ω_u is P -invariant. The space Ω_u is first considered in [HN95].

Proposition 4.4 *The action $\Omega_u \times \text{Int}(P) \rightarrow \Omega_u$ is open.*

Proof. Let $a \in \text{Int}(P)$. It is enough to show that $\Omega_u \text{Int}(P)a$ is open in Ω_u (See Theorem 4.3, [RS15]). We claim that

$$\Omega_u \text{Int}(P)a = \{A \in \Omega_u : A \cap \text{Int}(P)a \neq \emptyset\}$$

which will imply that $\Omega_u \text{Int}(P)a$ is open.

Let $A \in \Omega_u \text{Int}(P)a$. Then $A = Bba$ for some $B \in \Omega_u$ and $b \in \text{Int}(P)$. Since $e \in B$, it follows that $ba \in A$. Hence $A \cap \text{Int}(P)a$ is non-empty. Now suppose $A \in \Omega_u$ and $A \cap \text{Int}(P)a$ is non-empty. Choose $b \in \text{Int}(P)$ such that $ba \in A$. Since $P^{-1}A \subset A$, it follows that $P^{-1}ba \subset A$, equivalently $P^{-1} \subset Aa^{-1}b^{-1}$, and $P^{-1}Aa^{-1}b^{-1} \subset Aa^{-1}b^{-1}$. This proves that $B = Aa^{-1}b^{-1} \in \Omega_u$. Then $A = Bba \in \Omega_u \text{Int}(P)a$. This completes the proof. \square

We summarise a few facts regarding the space Ω_u in the following remark.

Remark 4.5 *Note the following.*

- (1) $\Omega_u \text{Int}(P)a = \{A \in \Omega_u : a \in \text{Int}(A)\}$. If $A \cap \text{Int}(P)a \neq \emptyset$ then $a \in \text{Int}(P)^{-1}A$ which is open and contained in A . Thus $a \in \text{Int}(A)$. Now suppose $a \in \text{Int}(A)$ then $\text{Int}(A) \cap Pa$ is non-empty. Since $\text{Int}(P)a$ is dense in Pa , it follows that $\text{Int}(A) \cap \text{Int}(P)a$ is non-empty and hence $A \cap \text{Int}(P)a$ is non-empty.

- (2) If $A \in \Omega_u$ then $\text{Int}(A)$ is dense in A and the boundary $\partial(A)$ has measure zero. This follows from Lemma 4.1 of [RS15]
- (3) Let $A \in \Omega_u$ and $g \in G$. Then $(A, g) \in \Omega_u \rtimes P$ if and only if $Ag \in \Omega_u$ if and only if $g^{-1} \in A$. We leave this verification to the reader.
- (4) The map $\Omega_u \ni A \rightarrow 1_A \in L^\infty(G)$ is continuous and injective and hence an embedding. Here $L^\infty(G)$ is given the weak *-topology. Let $\mathcal{G}_u := \Omega_u \rtimes P$. Then Proposition 4.4 implies that \mathcal{G}_u has a Haar system. Moreover a Haar system on \mathcal{G}_u is given by $(1_{Q_A}(g)dg)_{A \in \Omega_u}$. For $A \in \Omega_u$, observe that $Q_A := \{g \in G : (A, g) \in \mathcal{G}_u\}$ is A^{-1} .

By the definition of a Haar system, it follows that for $f \in C_c(\mathcal{G}_u)$, $\Omega_u \ni A \rightarrow \int f(x, g)1_A(g^{-1})dg$ is continuous. In particular, for $f \in C_c(G)$, the function $\Omega_u \ni A \rightarrow \int f(g)1_A(g^{-1})dg$ is continuous. As a consequence, the map $\Omega_u \ni A \rightarrow 1_A \in L^\infty(G)$ is continuous.

Suppose $A, B \in \Omega_u$ such that $1_A = 1_B$ in $L^\infty(G)$. Then $A \setminus B$ and $B \setminus A$ has measure zero. If $A \setminus B$ is non-empty then $\text{Int}(A) \setminus B$ is non-empty since $\text{Int}(A)$ is dense in A . But $\text{Int}(A) \setminus B$ is open and hence cannot have measure zero. Thus $A \setminus B = \emptyset$. Similarly $B \setminus A = \emptyset$. Hence $A = B$. This proves the map $\Omega_u \ni A \rightarrow 1_A \in L^\infty(G)$ is injective. Thus we can consider Ω_u as a compact subset of $L^\infty(G)$.

Let $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections. Denote the commutative C^* -algebra generated by $\{\int f(g)E_g dg : f \in L^1(G)\}$ by \mathcal{A} and let Ω be the spectrum of \mathcal{A} . For $f \in L^1(G)$, let $E_f := \int f(g)E_g dg$.

Proposition 4.6 *Let χ be a character of \mathcal{A} and let A be its support. Let $f \in C_c(G)$. Then*

- (1) $\chi\left(\int f(g)1_{A^c}(g)E_g dg\right) = 0$.
- (2) if $\text{supp}(f) \subset \text{Int}(A)$ then $\chi\left(\int f(g)E_g dg\right) = \int f(g)dg$, and
- (3) we have the equality $\chi\left(\int f(g)E_g dg\right) = \int f(g)1_A(g)dg$.

Proof. First observe that if $\text{supp}(f) \subset A^c$, where A^c denotes the complement of A , then $\chi(E_f) = 0$. This follows from the definition of A and by a partition of unity argument.

Now write $A^c = \bigcup_{n=1}^{\infty} K_n$ with K_n compact and K_n increasing. This is possible as A^c is open. Choose $\phi_n \in C_c(G)$ such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ on K_n and $\text{supp}(\phi_n) \subset A^c$.

Note that $\phi_n \rightarrow 1_{A^c}$ pointwise. Hence $f\phi_n$ converges to $f1_{A^c}$ in $L^1(G)$. This implies that $E_f\phi_n$ converges to $E_f1_{A^c}$. Since $\chi(E_{f\phi_n}) = 0$, it follows that $\chi(E_{f1_{A^c}}) = 0$. This proves (1).

Let $f \in C_c(G)$ be such that $\text{supp}(f) \subset \text{Int}(A)$. Let $g \in \text{supp}(f)$. Then $\text{Int}(A) \cap Pg$ is non-empty. Since $\text{Int}(P)$ is dense in P , it follows that $\text{Int}(A) \cap \text{Int}(P)g$ is non-empty. Let $s \in \text{Int}(P)$ be such that $sg \in \text{Int}(A)$. Then $(sg)g^{-1} \in \text{Int}(P)$. Since $\text{Int}(P)$ is open, we can choose open sets U and V contained in $\text{Int}(A)$, with compact closures, such that $(g, sg) \in U \times V \subset \text{Int}(A) \times \text{Int}(A)$ and $VU^{-1} \subset \text{Int}(P)$. Then by Proposition 3.3, it follows that for $g_1 \in V$ and $g_2 \in U$, $E_{g_1}E_{g_2} = E_{g_1}$.

Since $\text{supp}(f)$ is compact, it follows that there exists finitely many non-empty open sets $(U_i)_{i=1}^n$ and $(V_i)_{i=1}^n$ with compact closures, contained in $\text{Int}(A)$, such that $\text{supp}(f) \subset \bigcup_{i=1}^n U_i$ and $V_i U_i^{-1} \subset \text{Int}(P)$. A partition of unity argument allows us to write f as $f = \sum_{i=1}^n f_i$ with $\text{supp}(f_i) \subset U_i$. Thus to prove (2), it is enough to show $\chi(E_{f_i}) = \int f_i(g)dg$.

Since V_i is a non-empty open set contained in A , by Remark 4.2, it follows that there exists $\phi_i \in C_c(G)$ such that $\text{supp}(\phi_i) \subset V_i$ and $\chi(E_{\phi_i}) \neq 0$. Observe the following

$$\begin{aligned} E_{\phi_i}E_{f_i} &= \int_{V_i \times U_i} \phi_i(g_1)f_i(g_2)E_{g_1}E_{g_2}dg_1dg_2 \\ &= \int_{V_i \times U_i} \phi_i(g_1)f_i(g_2)E_{g_1}dg_1dg_2 \\ &= \left(\int f_i(g_2)dg_2 \right) \int \phi_i(g_1)E_{g_1}dg_1 \\ &= \left(\int f_i(g_2)dg_2 \right) E_{\phi_i}. \end{aligned}$$

Since χ is multiplicative, it follows that $\chi(E_{\phi_i})\chi(E_{f_i}) = \left(\int f_i(g)dg \right) \chi(E_{\phi_i})$. Now $\chi(E_{\phi_i}) \neq 0$. Hence $\chi(E_{f_i}) = \int f_i(g)dg$. This proves (2).

Now let $f \in C_c(G)$ be given. By (1), it follows that $\chi(E_f) = \chi(E_{f1_A})$. But since the boundary of A has measure zero, it follows that $1_{\text{Int}(A)} = 1_A$ a.e. Thus $\chi(E_f) = \chi(E_{f1_{\text{Int}(A)}})$. Write $\text{Int}(A) = \bigcup_n K_n$ with K_n compact and K_n increasing. Choose $\phi_n \in C_c(G)$ such that $\phi_n = 1$ on K_n and $\text{supp}(\phi_n) \subset \text{Int}(A)$. Then $\phi_n \rightarrow 1_{\text{Int}(A)}$ pointwise and hence $f\phi_n$ converges to $f1_{\text{Int}(A)}$ in $L^1(G)$. Note that $\text{supp}(f\phi_n) \subset \text{Int}(A)$.

Now calculate, as follows, to find that

$$\begin{aligned}
\chi(E_f) &= \chi(E_{f1_{Int(A)}}) \\
&= \lim_n \chi(E_{f\phi_n}) \\
&= \lim_n \int f(g)\phi_n(g)dg \quad (\text{by (2)}) \\
&= \int f(g)1_{Int(A)}(g)dg \\
&= \int f(g)1_A(g)dg \quad (\text{Since } 1_A = 1_{Int(A)} \text{ in } L^\infty(G)).
\end{aligned}$$

This proves (3). This completes the proof. \square .

Proposition 4.7 *For $\chi \in \Omega$, let A_χ be its support. Then the map $\Omega \ni \chi \rightarrow A_\chi \in \Omega_u$ is one-one, continuous and P -equivariant. Consequently, the action of P on Ω is injective.*

Proof. By Proposition 4.3, it follows that $A_\chi \in \Omega_u$ if $\chi \in \Omega$. For $f \in C_c(G)$, by Proposition 4.6, $\chi(E_f) = \int f(g)1_{A_\chi}(g)dg$. Hence the map $\Omega \ni \chi \rightarrow A_\chi \in L^\infty(G)$ is one-one and continuous where $L^\infty(G)$ is given the weak *-topology. By part (4) of Remark 4.5, it follows that $\Omega \ni \chi \rightarrow A_\chi \in \Omega_u$ is one-one and continuous.

Let $f \in C_c(G)$, $a \in P$ and $\chi \in \Omega$ and A be the support of χ . Observe that

$$\begin{aligned}
(\chi.a)\left(\int f(g)E_g dg\right) &= \chi\left(\int f(g)V_a^*E_gV_a dg\right) \\
&= \chi\left(\int f(g)E_{ga^{-1}} dg\right) \\
&= \chi\left(\int f(ga)\Delta(a)E_g dg\right) \\
&= \int f(ga)\Delta(a)1_A(g)dg \\
&= \int f(g)1_A(ga^{-1})dg \\
&= \int f(g)1_{Aa}(g)dg.
\end{aligned}$$

Hence the support of $\chi.a$ is Aa . Thus the map $\Omega \ni \chi \rightarrow A_\chi \in \Omega_u$ is a continuous P -equivariant embedding. This completes the proof. \square .

Thus we can and will consider Ω as a subset of Ω_u with the subspace topology.

5 Haar system on $\Omega \rtimes P$

In this section, we show that the semi-direct product $\Omega \rtimes P$ admits a Haar system. We prove that the action $\Omega \times \text{Int}(P) \rightarrow \Omega$ is open. To prove this, we need an analogue of Proposition 4.1 in the topological setting.

Proposition 5.1 *Let $a \in \text{Int}(P)$ and $A \in \Omega$. Then $a \in A$ if and only if $Aa^{-1} \in \Omega$.*

Proof. Let $a \in \text{Int}(P)$ and $A \in \Omega$ be given. Suppose $B := Aa^{-1} \in \Omega$. Since $e \in B$, it follows that $a \in A$. Now suppose $a \in A$. In addition, assume that $a \in \text{Int}(A)$. Let χ be the character defining A . Then for $f \in C_c(G)$,

$$\chi\left(\int f(g)E_g dg\right) = \int f(g)1_A(g)dg.$$

Choose a decreasing sequence of open sets (U_n) in G such that

- (1) the intersection $\bigcap_{n=1}^{\infty} U_n = \{a\}$,
- (2) if U is open in G and $a \in U$ then there exists N such that $U_n \subset U$ for $n \geq N$, and
- (3) for every n , $U_n \subset \text{Int}(P) \cap \text{Int}(A)$.

This is possible, for we can choose a metric and let (U_n) be the open balls containing a with $\text{diam}(U_n) \rightarrow 0$. For every $n \in \mathbb{N}$, choose $f_n \in C_c(G)$ such that $f_n \geq 0$, $\int f_n(g)dg = 1$ and $\text{supp}(f_n) \subset U_n$. Note that $\chi(E_{f_n}) = \int f_n(g)1_A(g)dg = 1$ since $\text{supp}(f_n) \subset \text{Int}(A)$.

Let ϕ_n be the linear functional on the commutative C^* -algebra \mathcal{A} defined by

$$\phi_n(T) = \chi\left(\int_{b \in P} f_n(b)V_b T V_b^* db\right).$$

Note that ϕ_n is well defined by Lemma 3.6 and is clearly positive. Note that

$$\begin{aligned} \phi_n(1) &= \chi\left(\int_{b \in \text{supp}(f_n)} f_n(b)E_b db\right) \\ &= \int_{b \in \text{supp}(f_n)} f_n(b)1_A(b)db \\ &= \int_{b \in \text{supp}(f_n)} f_n(b)db \quad (\text{Since } \text{supp}(f_n) \subset \text{Int}(A)) \\ &= 1. \end{aligned}$$

Thus ϕ_n is a state for every n . But the set of states on a unital C^* -algebra is weak*-compact. By choosing a subsequence if necessary we can assume without loss of generality that (ϕ_n) converges in the weak*-topology and let ϕ be its limit.

Recall that for $a \in P$, $\alpha_a : \mathcal{A} \rightarrow \mathcal{A}$ is given by $\alpha_a(T) = V_a^* T V_a$. By Proposition 4.7, it follows that for every $a \in \mathcal{A}$, α_a is surjective.

Claim: $\phi \circ \alpha_a = \chi$. It is enough to show that $(\phi \circ \alpha_a)(T) = \chi(T)$ for $T \in \tilde{\mathcal{A}}$. Let $T = E_\psi$ for some $\psi \in C_c(G_m)$.

Observe that for $n \in \mathbb{N}$,

$$\begin{aligned}
& \phi_n(\alpha_a(E_\psi)) \\
&= \chi\left(\int_{b \in P} f_n(b) V_b \alpha_a(E_\psi) V_b^* db\right) \\
&= \chi\left(\int_{b \in P} f_n(b) \psi(g_1, g_2, \dots, g_m) V_b V_a^* E_{g_1} \cdots E_{g_m} V_a V_b^* db dg_1 \cdots dg_m\right) \\
&= \chi\left(\int_{b \in P} f_n(b) \psi(g_1, g_2, \dots, g_m) E_b E_{g_1 a^{-1} b} \cdots E_{g_m a^{-1} b} db dg_1 \cdots dg_m\right) \\
&= \chi\left(\int_{b \in P} \Delta(b^{-1} a)^m f_n(b) \psi(g_1 b^{-1} a, g_2 b^{-1} a, \dots, g_m b^{-1} a) E_b E_{g_1} E_{g_2} \cdots E_{g_m} db dg_1 dg_2 \cdots dg_m\right).
\end{aligned}$$

Let $\epsilon > 0$ be given. Since ψ is continuous and compactly supported, it follows that there exists an open set U such that $a \in U$ and for $b \in U$ and $g_1, g_2, \dots, g_m \in G$,

$$\int |\Delta(b^{-1} a)^m \psi(g_1 b^{-1} a, g_2 b^{-1} a, \dots, g_m b^{-1} a) - \psi(g_1, g_2, \dots, g_m)| dg_1 dg_2 \cdots dg_m \leq \epsilon.$$

Choose $N \geq 1$ such that for $n \geq N$, $U_n \subset U$. Then for $n \geq N$, $\text{supp}(f_n) \subset U$. Note that for $n \geq N$,

$$\begin{aligned}
& |\phi_n(\alpha_a(E_\psi)) - \chi(E_\psi)| \\
&= |\phi_n(\alpha_a(E_\psi)) - \chi(E_{f_n} E_\psi)| \\
&\leq \int_{b \in U_n} f_n(b) \left(\int |\Delta(b^{-1} a)^m \psi(g_1 b^{-1} a, \dots, g_m b^{-1} a) - \psi(g_1, g_2, \dots, g_m)| dg_1 \cdots dg_m \right) db \\
&\leq \epsilon \int_{b \in U_n} f_n(b) \\
&\leq \epsilon.
\end{aligned}$$

Thus it follows that $\phi_n(\alpha_a(E_\psi)) \rightarrow \chi(E_\psi)$ and hence $\phi \circ \alpha_a = \chi$. This proves the claim.

Since α_a is surjective on \mathcal{A} , it follows that ϕ is a character of \mathcal{A} . Let $B \in \Omega$ be the support of ϕ . Then $\phi \circ \alpha_a = \chi$ translates to the equation $Ba = A$. Thus $Aa^{-1} \in \Omega$. Now suppose $a \in A$. Let (s_n) be a sequence in $\text{Int}(P)$ converging to the identity element e . Then $s_n^{-1}a \in \text{Int}(P)$ eventually, for $s_n^{-1}a \rightarrow a$ and $a \in \text{Int}(P)$. But $\text{Int}(P)^{-1}A \subset \text{Int}(A)$.

Hence $s_n^{-1}a \in \text{Int}(A)$. By what we have proved, it follows that $Aa^{-1}s_n \in \Omega$ eventually. However Ω is a compact subset of Ω_u and $(Aa^{-1}s_n)$ converges to Aa^{-1} . From this we conclude that $Aa^{-1} \in \Omega$. This completes the proof. \square

Just like in the discrete case, we have the following theorem.

Proposition 5.2 *Let $A \in \Omega$ and $g \in G$. Then $Ag \in \Omega$ if and only if $g^{-1} \in A$. Also the semi-direct product groupoid $\Omega \rtimes P$ has a Haar system.*

Proof. Let $g \in G$ and $A \in \Omega$ be given. Suppose $Ag \in \Omega$. Since $e \in Ag$, it follows that $g^{-1} \in A$. Now suppose $g^{-1} \in A$. As $G = (\text{Int}(P))(\text{Int}(P))^{-1}$, write $g = ab^{-1}$ with $a, b \in \text{Int}(P)$. Then $ba^{-1} \in A$ or $b \in Aa \in \Omega$. By Proposition 5.1, it follows that $Aab^{-1} \in \Omega$. Hence $Ag \in \Omega$.

To prove that $\Omega \rtimes P$ has a Haar system, it is enough to show that the action $\Omega \times P \rightarrow \Omega$ is open. By Theorem 4.3 of [RS15], it is enough to show that $\Omega \text{Int}(P)a$ is open in Ω for every $a \in P$. Let $a \in P$ be given.

Claim: $\Omega \text{Int}(P)a = \{A \in \Omega : A \cap \text{Int}(P)a \neq \emptyset\}$.

Suppose $A \cap \text{Int}(P)a$ is non-empty. Then there exists $s \in \text{Int}(P)$ such that $sa \in A$. By Proposition 5.1, $B := Aa^{-1}s^{-1} \in \Omega$. Thus $A = Bsa \in \Omega \text{Int}(P)a$. Suppose $A \in \Omega \text{Int}(P)a$. Then $A = Bsa$ for some $B \in \Omega$ and $s \in \text{Int}(P)$. Since $e \in B$, it follows that $sa \in A$. Thus $A \cap \text{Int}(P)a$ is non empty. This proves the claim.

The set $\{A \in \mathcal{C}(G) : A \cap \text{Int}(P)a \neq \emptyset\}$ is open in $\mathcal{C}(G)$, when $\mathcal{C}(G)$ is given the Vietoris topology. This implies that $\Omega \text{Int}(P)a$ is open in Ω . This completes the proof. \square

Remark 5.3 *Consider the groupoid $\Omega_u \rtimes P$. Then by Proposition 5.2 and Statement (3) of Remark 4.5, it follows that Ω is an invariant subset of Ω_u . Moreover the groupoid $\Omega \rtimes P$ is just the restriction $\Omega_u \rtimes P|_{\Omega}$.*

We end this section by describing Ω in the case of the Wiener-Hopf representation. Recall that the Wiener-Hopf representation $V : P \rightarrow B(L^2(P))$ is given by the formula: For $a \in P$ and $\xi \in L^2(P)$,

$$V_a(\xi)(x) := \begin{cases} \xi(xa^{-1})\Delta(a)^{\frac{-1}{2}} & \text{if } x \in Pa \\ 0 & \text{if } x \notin Pa. \end{cases}$$

Here Δ denotes the modular function of the group G . Note that for $g \in G$ and $\xi \in L^2(P)$, W_g is given by

$$W_g(\xi)(x) := \begin{cases} \xi(xg^{-1})\Delta(g)^{\frac{-1}{2}} & \text{if } xg^{-1} \in P \\ 0 & \text{if } xg^{-1} \notin P. \end{cases}$$

Let $M : L^\infty(P) \rightarrow B(L^2(P))$ be the multiplication representation. Observe that for $g \in G$, $E_g = W_g W_g^* = M(1_{P.g})$ where $P.g := \{xg : x \in P\} \cap P$. Denote the algebra of bounded continuous functions on P by $C_b(P)$. Since $\overline{Int(P)} = P$, it follows that M is a faithful representation of $C_b(P)$. For $f \in C_c(G)$, let $1_P * f \in C_b(P)$ be defined by

$$\begin{aligned} 1_P * f(t) &= \int 1_P(ts) f(s^{-1}) ds \\ &= \int 1_P(ts^{-1}) f(s) \Delta(s)^{-1} ds \\ &= \int 1_{P^{-1}t}(s) f(s) \Delta(s)^{-1} ds \end{aligned}$$

Observe that given $a \in P$, there exists $f \in C_c(G)$ such that $(1_P * f)(a) = 1$. For, if $a \in P$, choose $f \in C_c(G)$ such that $supp(f) \subset Int(P)^{-1}a$ and $\int f(s) \Delta(s)^{-1} ds = 1$. For such an f , $(1_P * f)(a) = 1$.

Now let $f \in C_c(G)$ and $\xi \in L^2(P)$ be given. Calculate as below to find that

$$\begin{aligned} < \left(\int f(g) E_g dg \right) \xi, \xi > &= \int f(g) < E_g \xi, \xi > dg \\ &= \int f(g) \left(\int_{x \in P} 1_{P.g}(x) |\xi(x)|^2 dx \right) dg \\ &= \int_{x \in P} \left(\int f(g) 1_{P.g}(x) dg \right) |\xi(x)|^2 dx \\ &= \int_{x \in P} \left(\int f(g) 1_P(xg^{-1}) dg \right) |\xi(x)|^2 dx \\ &= \int_{x \in P} \left(\int 1_P(xg) f(g^{-1}) \Delta(g)^{-1} dg \right) |\xi(x)|^2 dx \\ &= \int_{x \in P} (1_P * \hat{f})(x) |\xi(x)|^2 dx \\ &= < M(1_P * \hat{f}) \xi, \xi > . \end{aligned}$$

where $\hat{f}(g) = f(g) \Delta(g)$. Thus the C^* -algebra generated by $\{\int f(g) E_g dg : f \in C_c(G)\}$, is isomorphic to the C^* -subalgebra of $C_b(P)$ generated by $\{1_P * f : f \in C_c(G)\}$. Thus for $a \in P$, there exists a character χ_a of \mathcal{A} such that

$$\begin{aligned} \chi_a \left(\int f(g) E_g dg \right) &= (1_P * \hat{f})(a) \\ &= \int 1_{P^{-1}a}(s) \hat{f}(s) \Delta(s)^{-1} ds \\ &= \int 1_{P^{-1}a}(s) f(s) ds \end{aligned}$$

This implies that the support of χ_a is $P^{-1}a$. Also $\{\chi_a : a \in P\}$ separates the elements of $C_b(P)$ and hence those of \mathcal{A} . This implies that $\{P^{-1}a : a \in P\}$ is dense in Ω . As a consequence, it follows that Ω is the closure of $\{P^{-1}a : a \in P\}$ in the space of closed subsets of G w.r.t. the Vietoris topology. The 'compactification' Ω of P is called the Wiener-Hopf compactification and is considered in [MR82] and in [RS15].

6 Covariant representations

In this section, let X be a compact Hausdorff space and assume that P acts on X on the right injectively. Let $X_0 := X \text{Int}(P)$. We also assume that the semi-direct product $\mathcal{G} := X \rtimes P$ admits a Haar system. Let Y be a dilation of X , as explained in Section 1, on which the group G acts. For $y \in Y$, let $Q_y := \{g \in G : y.g \in X\}$. Recall that for $y \in Y$ and $g \in G$, $1_{Q_y}(g) = 1_X(yg)$ and $1_{\text{Int}(Q_y)}(g) = 1_{X_0}(yg)$. Also note that for every $y \in Y$, Q_y is closed and $Q_y P \subset Q_y$. Thus by Lemma 4.1 of [RS15], it follows that for every $y \in Y$, the boundary of Q_y has measure zero and $\overline{\text{Int}(Q_y)} = Q_y$.

To state the next lemma, we need to fix some notations. Let $a \in P$ and let (U_n) be a decreasing sequence of open subsets of G such that $\bigcap_{n=1}^{\infty} U_n = \{a\}$ and if U is open and contains a then $U_n \subset U$ eventually. Note that for every n , $U_n \cap Pa$ is non-empty. Hence $U_n \cap \text{Int}(P)a$ is non-empty for every n . Choose $f_n \in C_c(G)$ such that $f_n \geq 0$, $\int f_n(g)dg = 1$ and $\text{supp}(f_n) \subset U_n \cap \text{Int}(P)a$. For $x \in X$, let

$$F_n(x) = \int f_n(g)1_X(xg^{-1})dg.$$

Then $F_n \in C(X)$. The continuity of F_n follows from the fact that $(1_{Q_x}(g)dg)_{x \in X}$ is a Haar system on $X \rtimes P$. Observe that (F_n) is uniformly bounded.

Lemma 6.1 *The sequence F_n converges pointwise to 1_{X_0a} .*

Proof. Since for $x \in X$, $1_{Q_x} = 1_{\text{Int}(Q_x)}$ a.e., it follows that F_n is given by the equation

$$F_n(x) = \int f_n(g)1_{X_0}(xg^{-1})$$

for $x \in X$. Let $x \in X$. From $Q_x P \subset Q_x$, it is easily verifiable that $a^{-1} \in \text{Int}(Q_x)$ if and only if $\text{Int}(Q_x) \cap a^{-1}P^{-1}$ is non-empty. Suppose $a^{-1} \in \text{Int}(Q_x)$ i.e. $xa^{-1} \in X_0$.

Let $U := \{g \in G : xg^{-1} \in X_0\}$. Then U is open and contains a . Thus there exists N such that $n \geq N$ implies $\text{supp}(f_n) \subset U$ eventually. Then for $n \geq N$, $F_n(x) = \int f_n(g)dg =$

1. Now suppose $xa^{-1} \notin X_0$ i.e. $a^{-1} \notin \text{Int}(Q_x)$. Then $\text{Int}(Q_x)^{-1} \cap Pa$ is empty. Thus for $g \in Pa$, $xg^{-1} \notin X_0$. Since $\text{supp}(f_n) \subset Pa$, it follows that $F_n(x) = 0$. This proves that (F_n) converges pointwise to 1_{X_0a} . This completes the proof. \square

Lemma 6.2 *There exists a sequence (s_n) in $\text{Int}(P)$ such that $s_{n+1}^{-1}s_n \in \text{Int}(P)$ and (s_n) converges to the identity element e*

Proof. Let (U_n) be a countable base (of open sets) at e . We can assume that U_n is decreasing. Now $U_1 \cap P$ contains e and is non-empty. Since $\text{Int}(P)$ is dense in P , it follows that $U_1 \cap \text{Int}(P)$ is non-empty. Pick $s_1 \in U_1 \cap \text{Int}(P)$. Now suppose that s_1, s_2, \dots, s_n are chosen such that $s_k \in \text{Int}(P) \cap U_k$ for $1 \leq k \leq n$ and $s_{k+1}^{-1}s_k \in \text{Int}(P)$ for $1 \leq k \leq n-1$. Since $e \in s_n(\text{Int}(P))^{-1} \cap U_{n+1}$, it follows that $s_n \text{Int}(P)^{-1} \cap U_{n+1} \cap P$ is non-empty. But $\overline{\text{Int}(P)} = P$. Thus $s_n \text{Int}(P)^{-1} \cap U_{n+1} \cap \text{Int}(P)$ is non-empty. Let $s_{n+1} \in s_n \text{Int}(P)^{-1} \cap U_{n+1} \cap \text{Int}(P)$.

Then it is clear that the sequence (s_n) constructed as above converges to e and $s_{n+1}^{-1}s_n \in \text{Int}(P)$ for every n . This completes the proof. \square .

Consider a sequence (s_n) as in Lemma 6.2 converging to the identity e . Let $a \in \text{Int}(P)$ and set $t_n := s_n^{-1}a$. Then observe that $t_{n+1}t_n^{-1} \in \text{Int}(P)$. Since $a \in \text{Int}(P)$, $\text{Int}(P)$ is open and (t_n) converges to a , we can assume without loss of generality that $t_n \in \text{Int}(P)$ for every n . With this notation, we have the following lemma.

Lemma 6.3 *The sequence $(1_{X_0t_n})$ decreases pointwise to 1_{Xa} .*

Proof. Since $t_{n+1}t_n^{-1} \in \text{Int}(P)$, it follows that $X_0t_{n+1} \subset X_0t_n$ for every n . Since $at_n^{-1} \in \text{Int}(P)$, it follows that $Xa \subset X_0t_n$ for every n . Thus $Xa \subset \bigcap_{n=1}^{\infty} X_0t_n$. Now suppose $y \in X_0t_n$ for every n . Then $yt_n^{-1} \subset X_0$ for every n . Note that $(yt_n^{-1}) \rightarrow ya^{-1}$. Since the closure of X_0 in Y is X , it follows that $ya^{-1} \in X$. Hence $y \in Xa$. This proves that $Xa = \bigcap_{n=1}^{\infty} X_0t_n$. This completes the proof. \square

Let $B(X)$ be the space of bounded Borel measurable functions on X . For ϕ in $B(X)$ and $g \in G$, let $R_g(\phi)$ be defined by

$$R_g(\phi)(x) := \begin{cases} \phi(x.g) & \text{if } x.g \in X \\ 0 & \text{if } x.g \notin X \end{cases}$$

Then $R_g(\phi) \in B(X)$.

Definition 6.4 Let $\pi : C(X) \rightarrow B(\mathcal{H})$ be a unital $*$ -representation and $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections. Denote the extension of π to $B(X)$, obtained via the Riesz representation theory, by π itself [See [Arv02]]. For $g = ab^{-1}$, let $W_g := V_b^* V_a$. The pair (π, V) is said to be a covariant representation of (X, P) if for $\phi \in B(X)$,

$$W_g \pi(\phi) W_g^* = \pi(R_{g^{-1}}(\phi)).$$

Remark 6.5 Since $G = (Int(P))(Int(P))^{-1}$, it follows that (π, V) is a covariant representation if and only if $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ and $V_a \pi(\phi) V_a^* = \pi(R_{a^{-1}}(\phi))$ for $\phi \in B(X)$ and $a \in Int(P)$. We leave this verification to the reader.

We fix a few notations that will be useful for the rest of this section.

Notations: Let Y be the dilation of X , as explained in Section 1, on which G acts. Then $\mathcal{G} := X \rtimes P$ is a closed subset of $X \times G$ and also of $Y \rtimes G$. For $\phi \in C_c(Y)$, we let $\hat{\phi} \in C(X)$ be the restriction. Define $\pi_Y : C_c(Y) \rightarrow B(\mathcal{H})$ by $\pi_Y(\phi) = \pi(\hat{\phi})$. For $\phi \in C_c(Y)$ and $g \in G$, let $R_g(\phi) \in C_c(Y)$ be given by $R_g(\phi)(y) = \phi(y.g)$ for $y \in Y$.

For $\psi \in C_c(Y \rtimes G)$ and $g \in G$, let $\psi_g \in C_c(Y)$ be defined by $\psi_g(y) = \psi(y, g)$. For $\chi \in C_c(\mathcal{G})$ and $g \in G$, let $\chi_g \in B(X)$ be defined by $\chi_g(x) = \chi(x, g)$ if $(x, g) \in \mathcal{G}$ and $\chi_g(x) = 0$ if $(x, g) \notin \mathcal{G}$.

Let $\pi : C(X) \rightarrow B(\mathcal{H})$ be a unital $*$ -representation. For $\xi \in \mathcal{H}$, let $d\mu_{\xi, \xi}$ be the probability measure on X such that

$$\int \phi(x) d\mu_{\xi, \xi}(x) = \langle \pi(\phi)\xi, \xi \rangle \text{ for } \phi \in C(X).$$

The same equality holds for $\phi \in B(X)$.

Proposition 6.6 Let $\pi : C(X) \rightarrow B(\mathcal{H})$ be a unital $*$ -representation and $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections. Then the following are equivalent.

(1) The pair (π, V) is a covariant representation.

(2) For $a \in Int(P)$ and $\phi \in C(X)$, $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ and $\pi(1_{X_{0a}}) = E_a$.

Proof. For $a \in P$, let $\sigma_a : X \rightarrow X$ be the map sending $x \rightarrow xa$.

Suppose (π, V) is a covariant representation. Then the covariance relation implies that for $g \in G$, $E_g := W_g W_g^* = \pi(1_{Xg \cap X})$.

Let $f \in C_c(G)$. Set $F(x) = \int f(g)1_X(xg^{-1})dg$. Then $F \in C(X)$. Now calculate to find that

$$\begin{aligned}
\langle \pi(F)\xi, \xi \rangle &= \int F(x)d\mu_{\xi, \xi}(x) \\
&= \int f(g)1_{Xg}(x)dg d\mu_{\xi, \xi}(x) \\
&= \int f(g) \left(\int 1_{Xg}(x)d\mu_{\xi, \xi}(x) \right) dg \\
&= \int f(g) \langle \pi(1_{Xg \cap X})\xi, \xi \rangle dg \\
&= \int f(g) \langle E_g \xi, \xi \rangle dg.
\end{aligned}$$

Thus $\pi(F) = \int f(g)E_g$.

Let $a \in \text{Int}(P)$ be given. Choose a sequence (f_n) as in Lemma 6.1 and Let $F_n(x) := \int f_n(g)1_{Xg}(x)$ for $x \in X$. Note that F_n is uniformly bounded. By Lemma 6.1, it follows that F_n converges pointwise to 1_{X_0a} . On the other hand, we have $\pi(F_n) = \int f_n(g)E_g dg$. Since $g \rightarrow E_g$ is strongly continuous, it is easily verifiable that $\int f_n(g)E_g dg$ converges strongly to E_a . Hence $\pi(1_{X_0a}) = E_a$. Clearly by definition $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$. This proves (1) implies (2).

Now assume (2). The equality $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ for $a \in P$ and $\phi \in C(X)$ translates to the fact that for $a \in P$ and $\xi \in \mathcal{H}$, the push-forward measure $(\sigma_a)_*(\mu_{\xi, \xi}) = \mu_{V_a \xi, V_a \xi}$. Hence $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ for $a \in P$ and $\phi \in B(X)$. Now by Remark 6.5, it is enough to show that $V_a \pi(\phi) V_a^* = \pi(R_{a^{-1}}(\phi))$ for $a \in \text{Int}(P)$ and $\phi \in B(X)$. Now let $a \in \text{Int}(P)$ and $\phi \in B(X)$ be given. Then by assumption (2), we have $V_a^* \pi(R_{a^{-1}}(\phi)) V_a = \pi(\phi)$. Hence $V_a \pi(\phi) V_a^* = E_a \pi(R_{a^{-1}}(\phi)) E_a$. By the strong continuity of $g \rightarrow E_g$, by assumption (2) and Lemma 6.3, it follows that $\pi(1_{Xa}) = E_a$. Hence $V_a \pi(\phi) V_a^* = \pi(1_{Xa} R_{a^{-1}}(\phi)) = \pi(R_{a^{-1}}(\phi))$. This completes the proof. \square

Theorem 6.7 *Let X be a compact Hausdorff space on which P acts injectively. Let $\mathcal{G} := X \rtimes P$. Assume that \mathcal{G} has a Haar system. For $\phi \in C(X)$ and $f \in C_c(G)$, let $\phi \otimes f \in C_c(\mathcal{G})$ be defined by the equation $(\phi \otimes f)(x, g) = \phi(x)f(g)$. We denote $1 \otimes f$ by \tilde{f} .*

Let (π, V) be a covariant representation of (X, P) on a Hilbert space \mathcal{H} . Then there exists a representation $\lambda : C^(\mathcal{G}) \rightarrow B(\mathcal{H})$ such that*

- (1) *For $f \in C_c(G)$, $\lambda(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g) W_{g^{-1}} dg$. Here Δ is the modular function of the group G .*

(2) For $\phi \in C(X)$ and $f \in C_c(G)$, $\lambda(\phi \otimes f) = \pi(\phi)\lambda(\tilde{f})$.

Proof of Theorem 6.7. Let $\phi \in C_c(\mathcal{G})$. We claim that $G \ni g \rightarrow \pi(\phi_g)W_{g^{-1}} \in B(\mathcal{H})$ is strongly continuous. Let $\Phi \in C_c(Y \rtimes G)$ be an extension of ϕ . Since (π, V) is co-variant, it follows that $E_g = \pi(1_{Xg \cap X})$. Now observe that $\pi(\widehat{\Phi}_g)W_{g^{-1}} = \pi(\phi_g)W_{g^{-1}}$. For $\pi(\widehat{\Phi}_g)W_{g^{-1}} = \pi(\widehat{\Phi}_g)E_{g^{-1}}W_{g^{-1}} = \pi(\widehat{\Phi}_g 1_{Xg^{-1}})W_{g^{-1}} = \pi(\phi_g)W_{g^{-1}}$. But $g \rightarrow \widehat{\Phi}_g \in C(X)$ is continuous. Hence $G \ni g \rightarrow \pi(\widehat{\Phi}_g) \in B(\mathcal{H})$ is strongly continuous and consequently $G \ni g \rightarrow \pi(\phi_g)W_{g^{-1}} \in B(\mathcal{H})$ is strongly continuous.

For $\phi \in C_c(\mathcal{G})$, let $\lambda(\phi) \in B(\mathcal{H})$ be

$$\lambda(\phi) := \int \Delta(g)^{-\frac{1}{2}} \pi(\phi_g) W_{g^{-1}} dg.$$

Also we have shown that if $\Phi \in C_c(Y \rtimes G)$ is an extension of $\phi \in C_c(\mathcal{G})$, then

$$\lambda(\phi) = \int \Delta(g)^{-\frac{1}{2}} \pi_Y(\Phi_g) W_{g^{-1}} dg.$$

For $\phi \in C_c(\mathcal{G})$, calculate as follows to find that

$$\begin{aligned} \lambda(\phi)^* &= \int W_g \pi(\overline{\phi_g}) \Delta(g)^{-\frac{1}{2}} dg \\ &= \int W_g \pi(\overline{\phi_g}) W_g^* W_g \Delta(g)^{-\frac{1}{2}} dg \\ &= \int \pi(R_{g^{-1}}(\overline{\phi_g})) W_g \Delta(g)^{-\frac{1}{2}} dg \\ &= \int \pi(R_g(\overline{\phi_{g^{-1}}})) W_{g^{-1}} \Delta(g)^{\frac{1}{2}} \Delta(g)^{-1} dg \\ &= \int \pi((\phi^*)_g) W_{g^{-1}} \Delta(g)^{-\frac{1}{2}} dg \\ &= \lambda(\phi^*). \end{aligned}$$

Thus λ preserves the adjoint.

Now let $\phi, \psi \in C_c(\mathcal{G})$ be given and let $\Phi, \Psi \in C_c(Y \rtimes G)$ be extensions of ϕ and ψ respectively. Consider the function on $Y \rtimes G$ defined by the equation

$$\Phi \circ \Psi(y, g) = \int \Phi(y, h) \Psi(y.h, h^{-1}g) 1_X(y.h) dh.$$

A simple application of the dominated convergence theorem together with the fact that $1_X(y.h) = 1_{X_0}(y.h)$ a.e. for every y implies that $\Phi \circ \Psi$ is continuous. Clearly $\Phi \circ \Psi$ is compactly supported and is an extension of $\phi * \psi$.

Let $g \in G$ and $\xi \in \mathcal{H}$ be given. Then

$$\begin{aligned}
\langle \pi_Y((\Phi \circ \Psi)_g)\xi, \xi \rangle &= \int \Phi \circ \Psi(x, g) d\mu_{\xi, \xi}(x) \\
&= \int \left(\int \Phi(x, h) \Psi(x.h, h^{-1}g) 1_X(xh) dh \right) d\mu_{\xi, \xi}(x) \\
&= \int \left(\int \Phi(x, h) \Psi(x.h, h^{-1}g) 1_{Xh^{-1}}(x) d\mu_{\xi, \xi}(x) \right) dh \\
&= \int \langle \pi_Y(\Phi_h) \pi_Y(R_h(\Psi_{h^{-1}g})) \pi_Y(1_{Xh^{-1}}) \xi, \xi \rangle dh \\
&= \int \langle \pi_Y(\Phi_h) \pi_Y(R_h(\Psi_{h^{-1}g})) E_{h^{-1}} \xi, \xi \rangle dh
\end{aligned}$$

Thus for $g \in G$, $\pi_Y((\Phi \circ \Psi)_g) = \int \pi_Y(\Phi_h) \pi_Y(R_h(\Psi_{h^{-1}g})) E_{h^{-1}} dh$. Now calculate to find that

$$\begin{aligned}
\lambda(\phi)\lambda(\psi) &= \int \Delta(gh)^{-\frac{1}{2}} \pi_Y(\Phi_g) W_{g^{-1}} \pi_Y(\Psi_h) W_{h^{-1}} dg dh \\
&= \int \Delta(gh)^{-\frac{1}{2}} \pi_Y(\Phi_g) W_g^* \pi_Y(\Psi_h) W_g W_{g^{-1}} W_{h^{-1}} dg dh \\
&= \int \Delta(gh)^{-\frac{1}{2}} \pi_Y(\Phi_g) \pi_Y(R_g(\Psi_h)) W_{g^{-1}} W_{h^{-1}} dg dh \\
&= \int \Delta(gh)^{-\frac{1}{2}} \pi_Y(\Phi_g) \pi_Y(R_g(\Psi_h)) E_{g^{-1}} W_{h^{-1}g^{-1}} dg dh \quad [\text{by Proposition 3.3}] \\
&= \int \left(\int \Delta(k)^{-\frac{1}{2}} \pi_Y(\Phi_g) \pi_Y(R_g(\Psi_{g^{-1}k})) E_{g^{-1}} W_{k^{-1}} dk \right) dg \\
&= \int \left(\int \pi_Y(\Phi_g) \pi_Y(R_g(\Psi_{g^{-1}k}) E_{g^{-1}} dg \right) \Delta(k)^{-\frac{1}{2}} W_{k^{-1}} dk \\
&= \int \Delta(k)^{-\frac{1}{2}} \pi_Y((\Phi \circ \Psi)_k) W_{k^{-1}} dk \\
&= \lambda(\phi * \psi).
\end{aligned}$$

Hence λ preserves the multiplication.

For $\phi \in B(X)$, one has $\|\pi(\phi)\| \leq \|\phi\|_\infty$ where $\|\cdot\|_\infty$ is the sup norm on $B(X)$. Let

K be a compact subset of G . Then for $\phi \in C_c(\mathcal{G})$ with $\text{supp}(\phi) \subset X \times K$, observe that

$$\begin{aligned} \|\lambda(\phi)\| &\leq \int \Delta(g)^{-\frac{1}{2}} \|\pi(\phi_g)\| dg \\ &\leq \int_{g \in K} \Delta(g)^{-\frac{1}{2}} \|\phi_g\| dg \\ &\leq \left(\sup_{g \in K} \Delta(g)^{-\frac{1}{2}} \right) \|\phi\|_\infty \int 1_K(g) dg. \end{aligned}$$

Thus it is clear that the map $\lambda : C_c(\mathcal{G}) \rightarrow B(\mathcal{H})$ is continuous when $C_c(\mathcal{G})$ is given the inductive limit topology and $B(\mathcal{H})$ is given the norm topology. By Renault's disintegration theorem, one obtains a bonafide representation $\lambda : C^*(\mathcal{G}) \rightarrow B(\mathcal{H})$. Conditions (1) and (2) follows just from definitions. This completes the proof. \square

7 The main theorem

Let $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections. Let \mathcal{A} and Ω be as in sections 3-5. Denote the open set $\Omega \text{Int}(P)$ by Ω_0 . Let $\pi : C(\Omega) \rightarrow B(\mathcal{H})$ be the representation induced by the inclusion $\mathcal{A} \subset B(\mathcal{H})$. Denote the extension to $B(\Omega)$ by π itself. First let us show that (π, V) is a covariant representation.

Lemma 7.1 *The pair (π, V) is a covariant representation.*

Proof. By definition, it follows that for $a \in P$, $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ for $\phi \in C(X)$. Now fix $a \in P$. Choose a sequence $(f_n) \in C_c(G)$ as in Lemma 6.1. Set $F_n := \int f_n(g) E_g dg \in C(\Omega)$. Then by the strong continuity of $g \rightarrow E_g$, it is clear that F_n converges strongly to E_a . Now by definition, for $A \in \Omega$, $F_n(A) = \int f_n(g) 1_A(g) dg$. By Proposition 5.2, it follows that $F_n(A) = \int f_n(g) 1_\Omega(Ag^{-1}) dg$ for $A \in \Omega$. By Lemma 6.1, it follows that F_n converges pointwise to $1_{\Omega_0 a}$. Hence $\pi(1_{\Omega_0 a}) = E_a$. The proof follows now from Proposition 6.6. \square

Proposition 7.2 *Let \mathcal{H} be a Hilbert space and $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections. Let Ω be as in Sections 3-5. Then there exists a $*$ -homomorphism $\lambda : C^*(\Omega \rtimes P) \rightarrow B(\mathcal{H})$ such that for $f \in C_c(G)$,*

$$\lambda(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g) W_{g^{-1}} dg.$$

Moreover the range of λ is generated by $\{\int f(g) W_g dg : f \in C_c(G)\}$.

Proof. For $f \in C_c(G)$, let $\widehat{f} \in C(\Omega)$ be defined by

$$\widehat{f}(A) := \int f(g)1_\Omega(Ag)dg = \int f(g)1_A(g^{-1}).$$

By (4) of Remark 4.5 and the fact that $\Omega \subset \Omega_u$, it follows that $\{\widehat{f} : f \in C_c(G)\}$ separates points of Ω . Thus by Proposition 2.1, the $*$ -algebra generated by $\{\widehat{f} : f \in C_c(G)\}$ is dense in $C^*(\Omega \rtimes P)$. Now the proof follows directly from Lemma 7.1 and Proposition 6.7. \square

Theorem 7.3 *Let \mathcal{H} be a Hilbert space and $V : P \rightarrow B(\mathcal{H})$ be an isometric representation with commuting range projections. Let $\Omega_u := \{A \in \mathcal{C}(G) : P^{-1} \subset A \text{ and } P^{-1}A \subset A\}$ with the Vietoris topology. Consider the right action of P on Ω_u by right multiplication. Then there exists $*$ -homomorphism $\lambda : C^*(\Omega_u \rtimes P) \rightarrow B(\mathcal{H})$ such that for $f \in C_c(G)$,*

$$\lambda(\widetilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g) W_{g^{-1}} dg.$$

Moreover the range of λ is generated by $\{\int f(g) W_g dg : f \in C_c(G)\}$.

Proof. By Remark 5.3, it follows that $\Omega \rtimes P$ is isomorphic to the restriction $\Omega_u \rtimes P|_\Omega$ and Ω is an invariant subset of Ω_u . Consider the natural map $\text{res} : C_c(\Omega_u \rtimes P) \rightarrow C_c(\Omega \rtimes P)$ which on $C_c(\Omega_u \rtimes P)$ is simply the restriction. Let $\widetilde{\lambda} : C^*(\Omega \rtimes P) \rightarrow B(\mathcal{H})$ be the representation as in Proposition 7.2. Now one completes the proof by setting $\lambda := \widetilde{\lambda} \circ \text{res}$. \square

Remark 7.4 *Proposition 7.3 says that the C^* -algebra of the groupoid $\Omega_u \rtimes P$ can be interpreted as the 'universal' C^* -algebra which encodes the isometric representations with commuting range projections. However, the space Ω_u is quite large to describe explicitly even for the simple example of the quarter plane $[0, \infty) \times [0, \infty) \subset \mathbb{R}^2$.*

We end this article by considering two well-known results which are a part of folklore in operator algebras.

Example 7.5 *Let $P := \mathbb{N}$ and $G := \mathbb{Z}$ with the discrete topology. Consider the one-point compactification $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$. The semigroup \mathbb{N} acts on \mathbb{N}_∞ by translation with the convention that $\infty + n = \infty$ for $n \in \mathbb{N}$. It is easy to verify that the map $\mathbb{N}_\infty \ni n \rightarrow (-\infty, n] \in \Omega_u$ is an \mathbb{N} -equivariant homeomorphism. Here $(-\infty, \infty]$ is just \mathbb{N} . The groupoid $\mathbb{N}_\infty \rtimes \mathbb{N}$ is amenable and $C_{\text{red}}^*(\mathbb{N}_\infty \rtimes \mathbb{N})$ is just the Toeplitz-algebra. Now Theorem 7.3 is just the well-known Coburn's theorem.*

Example 7.6 Let $\mathbb{R}_+ = [0, \infty)$. Let $P := \mathbb{R}_+$ and $G := \mathbb{R}$ with the usual Euclidean topology and addition as the group operation. Consider the one-point compactification $[0, \infty] := [0, \infty) \cup \{\infty\}$. The semigroup $[0, \infty)$ acts on $\mathbb{R} \cup \{\infty\}$ by translation with the convention that $\infty + x = \infty$ for $x \in [0, \infty)$. It is easily verifiable that the map $[0, \infty] \ni x \rightarrow (-\infty, x] \in \Omega_u$ is a \mathbb{R}_+ -equivariant homeomorphism. The groupoid $[0, \infty] \rtimes \mathbb{R}_+$ is amenable and $C_{red}^*([0, \infty] \rtimes [0, \infty))$ is the usual Wiener-Hopf algebra. [See [MR82]]

Observe that if $V : \mathbb{R}_+ \rightarrow B(\mathcal{H})$ is an isometric representation then the range projections $\{E_t := V_t V_t^* : t \geq 0\}$ commutes. For if $t = r + s$ then $E_t E_r = V_r V_s V_s^* V_r^* V_r V_r^* = E_t$. Hence if $t > r$, then $E_t E_r = E_t$. Now the claim follows from the fact that \mathbb{R}_+ is totally ordered. Thus if $V : \mathbb{R}_+ \rightarrow B(\mathcal{H})$ is an isometric representation, then there exists a representation $\pi : \mathcal{W}([0, \infty), \mathbb{R}) \rightarrow B(\mathcal{H})$ such that

$$\pi(\tilde{f}) = \int_0^\infty f(t) V_t + \int_{-\infty}^0 f(t) V_t^*$$

for $f \in C_c(\mathbb{R})$.

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S. SUNDAR (sundarsobers@gmail.com)

Chennai Mathematical Institute, H1 Sipcot IT Park,
Siruseri, Padur, 603103, Tamilnadu, INDIA.